Can one hear Shape?

Martin Reuter

reuter@mit.edu, Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139

The question "Can one hear the shape of a drum" has been asked in several contexts before (e.g., by Bers and Kac). It is a pictorial way of asking if the eigenvalues of the Laplacian on a given domain completely characterize its shape, in other words, if the spectrum is a complete shape descriptor (which it is not in general). Since the spectrum contains geometrical information and since it is an isometry invariant and therefore independent of the object's representation, parametrization, spatial position, and optionally of its size, it is well suited to be used as a fingerprint (Shape-DNA) in contemporary computer graphics applications like database retrieval, quality assessment, and shape matching in fields like CAD, engineering or medicine. We will explain why isometry invariance is so important and point out future directions of research.

The question "Can one hear the Shape of a Drum?" first formulated by Bers (see Protter [1]) and later published by Kac in 1966 [2] asks if the shape of a planar region is determined by the spectrum of the Laplacian. In fact, the idea to connect the eigenvalues (the spectrum) with geometric entities dates back to Weyl 1911 [3] who showed that the asympotic behavior of the eigenvalues depends on the surface area of the drum (or on the volume of a 3D solid). Of course the natural frequencies of a drum depend on its shape. These frequencies (square roots of the eigenvalues of the Laplacian) can be numerically computed if the shape is known. In his work, Kac proved that no other shape has the same spectrum as a disk (this is actually true for the n-dimensional ball). So the question if the shape of a drum can be computed from its spectrum had to be answered. For planar shapes this was finally done by Gorden, Webb and Wolpert in 1992 [4] who constructed pairs of concave polygons (containing corners) that are not congruent but isospectral. Nevertheless, these twins (no one has constructed a triple so far) have to share geometric and topologic features such as area, boundary length and number of holes (none), since these values are in fact spectrally determined (see McKean and Singer [5] who analyzed the heat trace expansion for a more general setting on a Riemannian manifold and Reuter [6] who first extracted these values numerically from the beginning sequence of different spectra of 2D and 3D manifolds).

So how can these spectra be employed for shape matching of geometric objects? Complex geometric objects have gained much importance in many different fields like computer aided design, engineering or in medical applications. Especially technologies to digitize 2D surfaces or 3D volumetric data (e.g. MRI scanners) opened up a new direction of research: away from the 2D images towards understanding and handling of 3D shapes. It is a basic question to identify, compare and recognize the shape of 2D surfaces and 3D solid objects given in many different representations. Shape analysis and shape matching (the process of determining the similarity between electronic models of objects) are subjects of a huge body of research. Basically all methods consist of two steps: First a signature or shape descriptor (SD) is computed. This step simplifies the shape by representing it in a form that can be handled and compared easily. If a shape descriptor cannot be made invariant with respect to position, size or isometry a prior alignment of the objects (registration/normalization) is necessary. In the second step the shape descriptors are compared (distance computation) using different similarity metrics defined on the space of the SD (e.g. Euclidean, Hausdorff, correlation).

The begining sequence of the Laplace spectra contains only positive real numbers and yields such a shape descriptor with many desirable properties. Though this descriptor is relatively insensitive to noise, it can still detect local shape differences. It can be compared easily and can be computed for many different shape representations. It can deal with objects containing cavities, depends continuously on shape deformations and can be made scaling invariant. Most importantly it is **isometry invariant**. Therefore we do not need any alignment of the objects (translation, rotation) and can even identify (almost) identical objects embedded differently in space. Examples are a hand with different finger positions, the same face with different expressions, a tree bending in the wind, a person in different body positions etc. Many objects in our world are not rigid. It is therefore of huge interest, to identify them in different poses.

Reuter, Wolter and Peinecke [6, 7, 8, 9] introduced a new spectral method for shape matching. The method uses the possibly normed beginning sequence of the Laplace-Beltrami spectrum of a Riemannian manifold (mainly surface or solid) as a shape descriptor (called "ShapeDNA"). The eigenvalues λ and eigenfunctions u are the solution of the Laplacian eigenvalue problem $\Delta u = -\lambda u$, where $\Delta u := \mathrm{div}(\mathrm{grad}\ u)$ with grad being the gradient and div the divergence with respect to the underlying domain or Riemannian manifold in general. The first smallest n corresponding eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ are taken as a shape descriptor. It should be noted that many versions of discrete graph and mesh Laplacians are know in the computer graphics comunity. The main difference of the Laplace-Beltrami spectrum computations (first presented in [7] using cubic FEM) is that it converges to the exact solution with high order accuracy.

The fact that the presented Laplace-Beltrami spectra are isometry invariants can be seen nicely in Figure 1, where the first 30 eigenvalues (surface spectra) are plotted for a female body shape in four different body positions and for the sphere (all

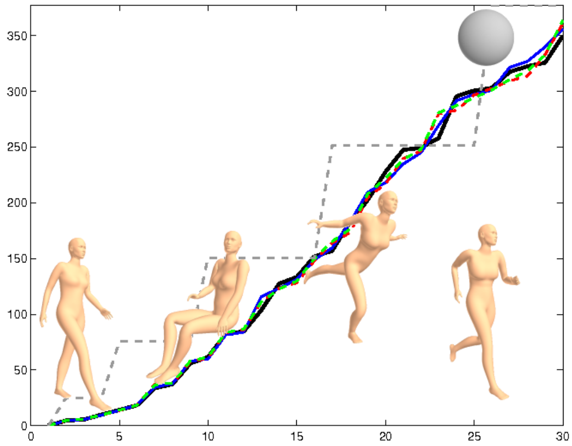

Fig. 1 Spectra are aligned for near isometric shapes

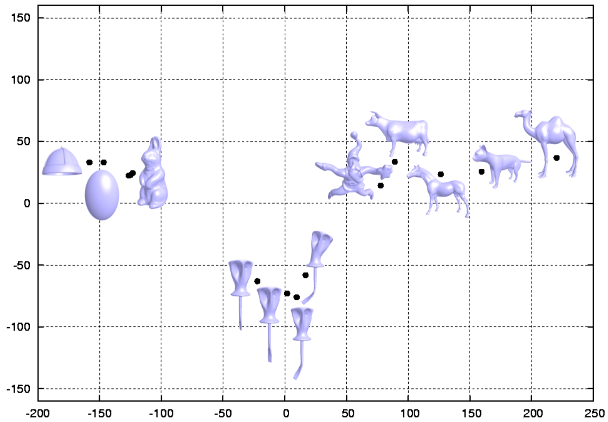

Fig. 2 Clustering of shapes with spherical topology

spectra are area normalized, the position of the shpere and the female body shapes in the plot are irrelevant). The spectra of the body shapes are closely aligned while the spectrum of the sphere differs a lot. It was presented in Reuter [6] that the ShapeDNA can be successfully applied for database retrieval, copyright protection and quality assessment. See Figure 2 for a plot presenting the first two principal components of the high dimensional ShapeDNA of a few closed surfaces. It can be seen how the ShapeDNA clusters the objects into meaningful groups (considering that the surface of the helmet is in fact a deformed ellipsoid where one cap has been flip to the inside). To distinguish the helmet from the ellipsoid it makes sense not to look at the spectra of their (almost isometric) boundary surfaces but instead at the spectra of their solid 3D bodies, which are very different from each other.

Recent work applies the "ShapeDNA" method to medical data [10, 11]. Shape analysis is important for medical image analysis to assess morphological changes that go beyond volume differences. Since changes of interest for medical applications (e.g., for the assessment of brain changes in schizophrenia research) are frequently subtle, but may be investigated through population studies, robust shape descriptors such as the ShapeDNA need to be combined with procedures for statistical hypothesis testing. We have shown statistically significant differences in shape (using such a methodology) between the caudate nuclei (a subcortical gray matter structure of the human brain involved in memory function, emotion processing, and learning) of patients diagnosed with schizoptypal personality disorder and the caudate nuclei of a normal control population. In particular, the study showed that in addition to significant variations in volume and surface area (already known before) real shape differences exist. In [11] the method has been extended to work directly on the voxel data obtained from MRI after segmentation. Different boundary conditions were analyzed (Neumann and Dirichlet) showing that the Neumann spectrum picks up shape differences already on a very coarse grid, making it better suited for 3D solid matching.

Fig. 3 EF 17 in different positions

Future research will deal with the eigenfunctions, that are attached locally to the geometry. It is of course far easier to compare a vector of numbers than functions, but the additional (localized) information contained in the eigenfunctions can help to register near isometric shapes (which can be achieved by analyzing the topology of the functions with the help of size theory or Morse theory). See Figure 3 showing the eigenfunction number 17 with nodal lines for two different body positions.

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